EUCLIDEAN PROOFS OF PROJECTIVE GEOMETRY THEOREMS

by

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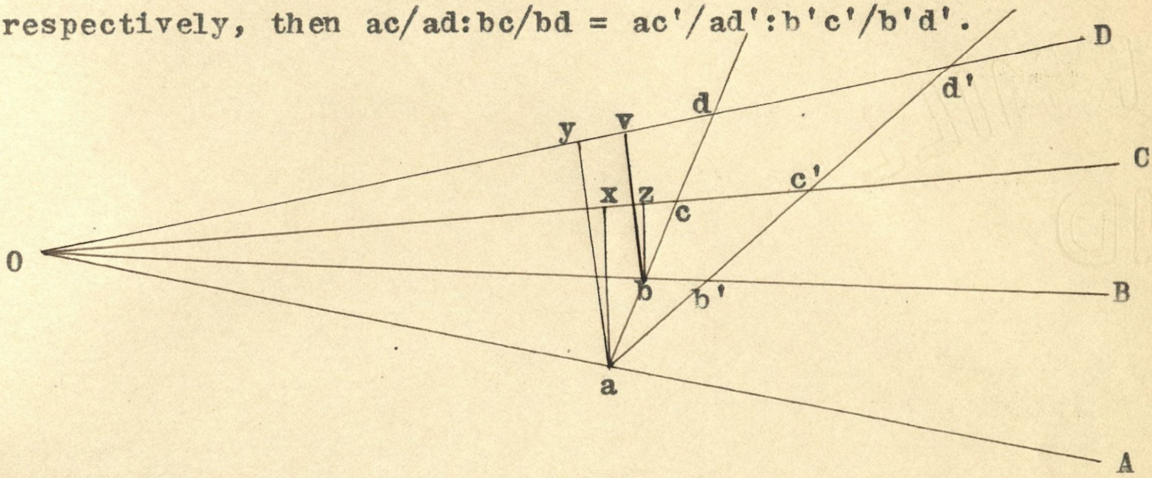
Many of the geometrical configurations and theorems of Projective Geometry are usually considered as being entirely without the realm of Euclidean Geometry. The purpose of this discussion is to present such proofs for some of these propositions that the student of elementary geometry may add them to his field of study. To this end, the following proofs are based on simple, fundamental principles of elementary Euclidean Geometry.

THEOREM 1. If the lines, ad and ad', cut the four lines, OA, OB, OC and OD, in the points, a,b,c,d, and a,b',c',d', respectively, then ac/ad:bc/bd = ac'/ad':b'c'/b'd'.



Proof: Draw ay and by perpendicular to OD, and ax and bz perpendicular to OC.

Let m =	ratio	ax/aO	Let n =	ratio	ax/ac
" r =	"	ay/aO	" s =	"	ay/ad
" t =	"	bz/bO	" u =	"	bv/bO
" e =	"	bz/bc	" g =	"	bv/bd

Then, by division, m/n = ac/aO and r/s = ad/aO
By division again, m/r = n/s.ac/ad (1)

Similarly, t/e = bc/bO and u/g = bd/bO
Dividing, t/u = e/g.bc/bd (2)
Triangles yad and vbd are similar (Angles resp. =)
Therefore, g = bv/bd = ay/ad = s (Sides proportional)
Likewise, triangles axc and bzc are similar and
e = bz/bc = ax/ac = n
Substituting in (2), t/u = n/s.bc/bd (3)

Dividing (1) by (3), we have:
m/r:t/u::ac/ad:bc/bd

Using ab'c'd' in the same way that we used abcd,
we have: m/r:t/u:: a'c'/ad' : b'c'/b'd'

From which, ac/ad:bc/bd = ac'/ad':b'c'/b'd'
(Equal to same thing: therefore equal to each other.)

DEFINITION: If a,b,c,d are four points on any given line, then the expression, ac/ad:bc/bd, is called the cross ratio of the points on that line.

The symbol, R, will be used in this discussion to indicate cross ratio. R(a,b,c,d) will be used to mean the cross ratio of the points , a,b,c,d.

COROLLARY 1. R(a,b,c,d) = R(b,a,d,c)

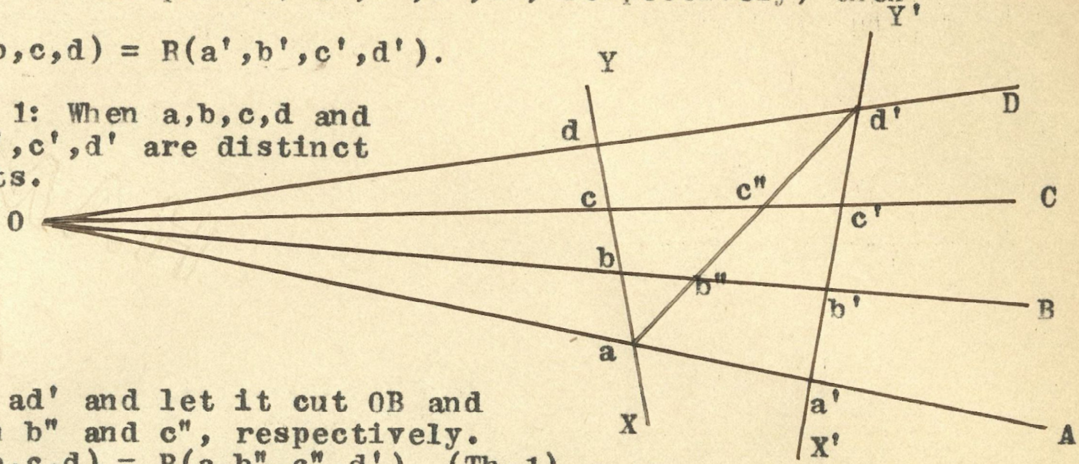
Proof: R(a,b,c,d) = ac/ad:bc/bd = ac.bd/ad.bc
R(b,a,d,c) = bd/bc:ad/ac = ac.bd/ad.bc
Therefore R(a,b,c,d) = R(b,a,d,c)
(Equal to same expression)

COROLLARY 2. R(a,b,c,d) = R(d,c,b,a)

Proof: R(a,b,c,d) = ac/ad:bc/bd = ac.bd/ad.bc
R(d,c,b,a) = db/da:cb/ca = db.ca/da.cb
Therefore R(a,b,c,d) = R(d,c,b,a)
(Equal to same expression)

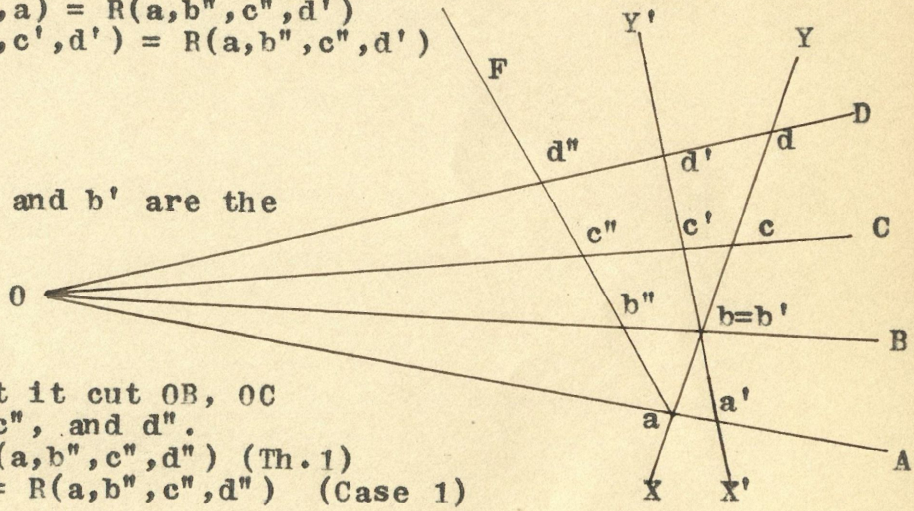
COROLLARY 3. If XY cuts the lines, OA, OB, OC and OD, in the points a,b,c,d, respectively, and X'Y' cuts the same lines in the points, a',b',c',d', respectively, then R(a,b,c,d) = R(a',b',c',d').

Case 1: When a,b,c,d and a',b',c',d' are distinct points.



Draw ad' and let it cut OB and OC in b" and c", respectively.
R(a,b,c,d) = R(a,b",c",d') (Th.1)
R(d',c',b',a') = R(d',c",b",a) (Th.1)
But R(d',c',b',a') = R(a',b',c',d') (Cor. 2)
And R(d',c",b",a) = R(a,b",c",d')
Hence: R(a',b',c',d') = R(a,b",c",d')
= R(a,b,c,d).

Case 2: When b and b' are the same point.



Draw aF and let it cut OB, OC and OD in b", c", and d".
R(a,b,c,d) = R(a,b",c",d") (Th.1)
R(a',b,c',d') = R(a,b",c",d") (Case 1)

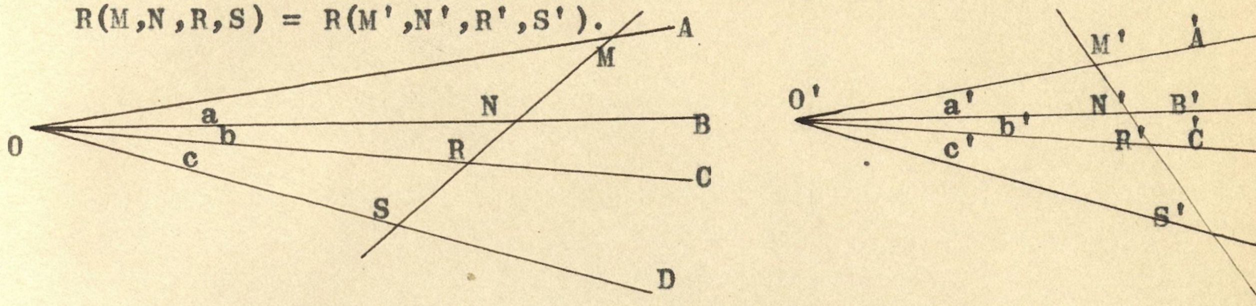
Therefore, R(a,b,c,d) = R(a',b,c',d')
(Equal to same cross ratio)

Case 3: When c and c' are identical.

(Proof same as in Case 2.)

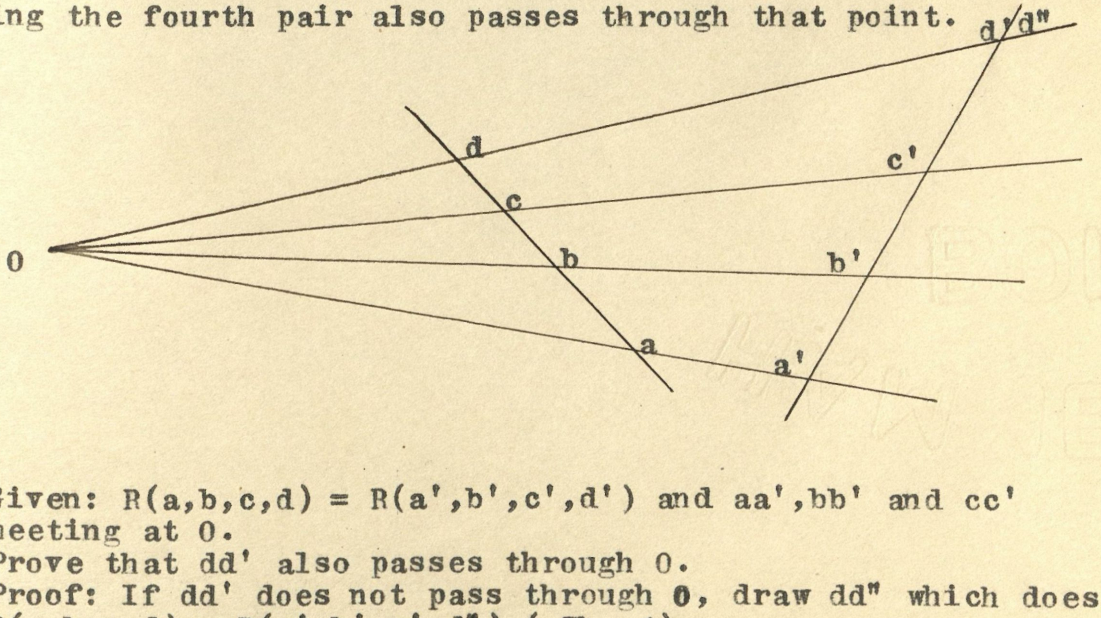
COROLLARY 4. If the figures, O-ABCD and O'-A'B'C'D' have the angles a,b,c, equal respectively to angles, a',b',c', and SM and S'M' are any transversals, then

$R(M,N,R,S) = R(M',N',R',S')$.



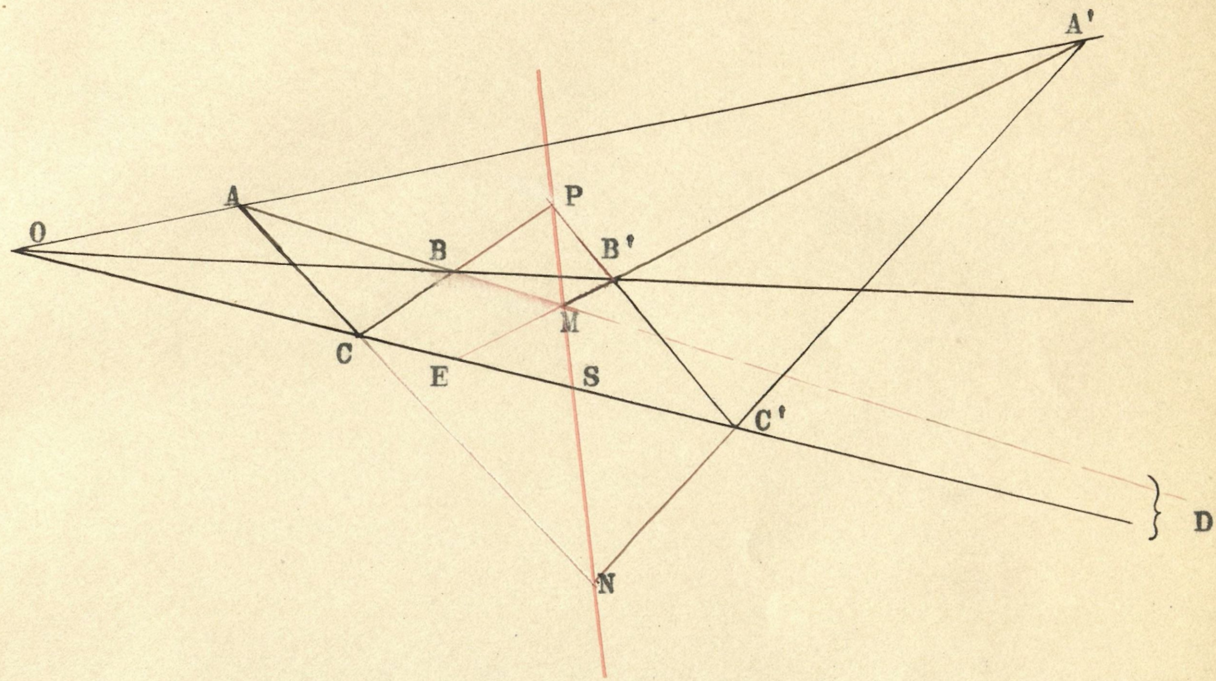
Proof: Since the angles of the two figures are equal respectively, the figures can be superimposed and they will coincide throughout excepting in general the transversals. Hence, $R(M,N,R,S) = R(M',N',R',S')$ by Corollary 3.

THEOREM 2. If two sets of points, a,b,c,d, and a',b',c',d', have the same cross ratio and the lines joining three pairs of corresponding points meet at a point, then the line joining the fourth pair also passes through that point.



Given: $R(a,b,c,d) = R(a',b',c',d')$ and aa',bb' and cc' meeting at O.
Prove that dd' also passes through O.
Proof: If dd' does not pass through O, draw dd" which does.
$R(a,b,c,d) = R(a',b',c',d")$ (Th. 1)
But $R(a,b,c,d) = R(a',b',c',d')$ by hypothesis.
Hence d' = d" and dd' passes through O.

THEOREM 3. If two triangles, ABC and A'B'C', have AB and A'B' meeting at M, BC and B'C' meeting at P and AC and A'C' meeting at N, when M,P and N are collinear points, then AA', BB' and CC' meet at a point, O.

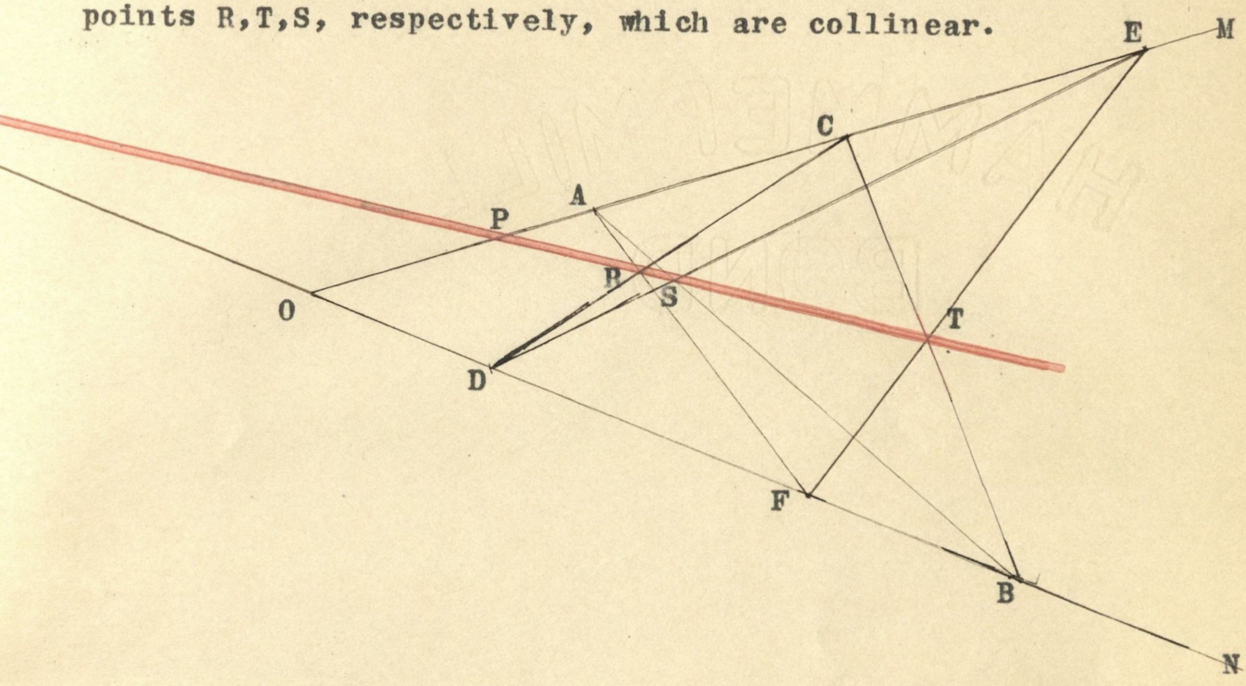


Proof: R(M,A',E,B') = R(M,N,S,P)   (Cor. 3, Th. 1)
R(M,A,D,B) = R(M,N,S,P)   " " " "

Therefore R(M,A',E,B') = R(M,A,D,B) (Equal to same ratio.)

Whence AA', BB' and ED (=CC') meet at a point (Th.2)

THEOREM 4. If A,C,E are any three points of a line OM and D,F,B any three points of ON, the three pairs of lines, AF and DC, CB and FE, AB and DE, meet in the points R,T,S, respectively, which are collinear.



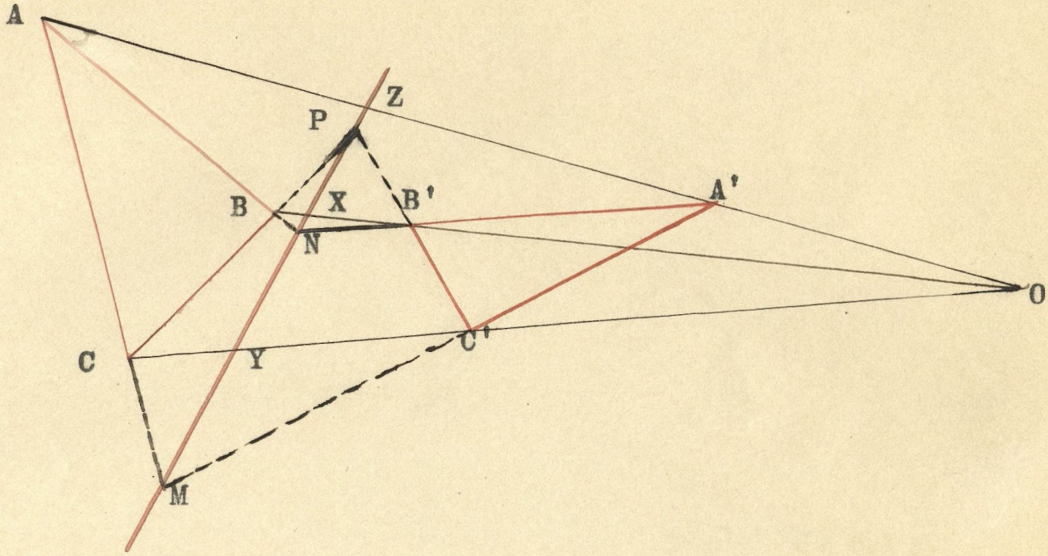
(In order to facilitate the reading of these proofs, the centers of radiating lines on which equal cross ratios are determined will be mentioned as occasion requires.)

Proof: Draw RS and let it cut OM and ON at P and V, respectively. From D, R(P,O,C,E) = R(P,V,R,S)
" A, R(V,O,B,F) = R(V,P,S,R) (Cor. to Th.1)
But R(P,V,R,S) = R(V,P,S,R) (Cor.1, Th.1)

Therefore R(P,O,C,E) = R(V,O,B,F) (Equal to same)
Hence CB, EF and RS meet at a point and the points R, S and T are collinear. (Th.2)

Note: The above figure is called the Configuration of Pappus.

THEOREM 5. (Converse of Theorem 3.) If the triangles ABC and A'B'C' have AA', BB' and CC' meeting at a point O, then P, N and M, the intersections of BC and B'C', AB and A'B', AC and A'C', respectively, are collinear.



Proof:

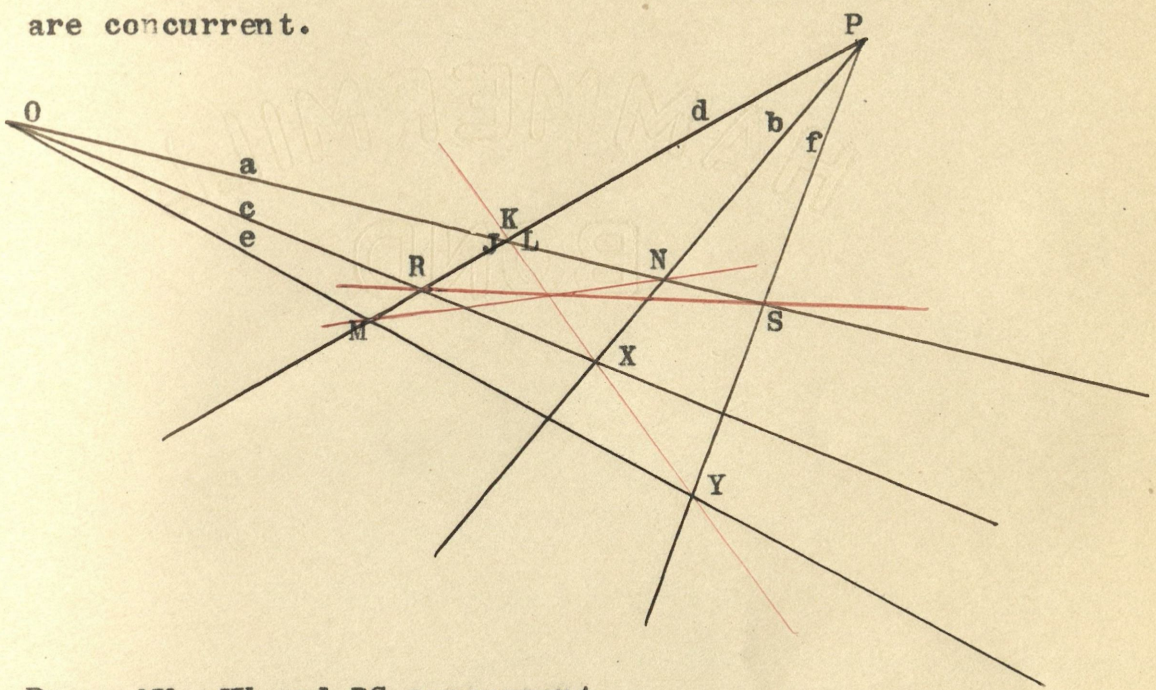
Draw MN. Let it cut OB, OC and OA in X, Y and Z, respectively.

From N, $R(B,X,B',O) = R(A,Z,A',O)$
" M, $R(C,Y,C',O) = R(A,Z,A',O)$ (Th. 1 & Cor.)

TTherefore $R(B,Z,B',O) = R(C,Y,C',O)$ (Equal to same)

Hence, BC, XY and B'C' meet at a point by Theorem 2 and the intersection of BC and B'C' is on MN.

THEOREM 6. If a,c,e, and d,b,f, are any three lines through O and P respectively, the lines joining the pairs of intersections, ab and ed; cd and af; and cb and ef, respectively, are concurrent.



Prove MN, XY and RS concurrent.
(OS, PM and XY are not necessarily concurrent. J is the intersection of OS and PM, L of OS and XY, and K of PM and XY.)

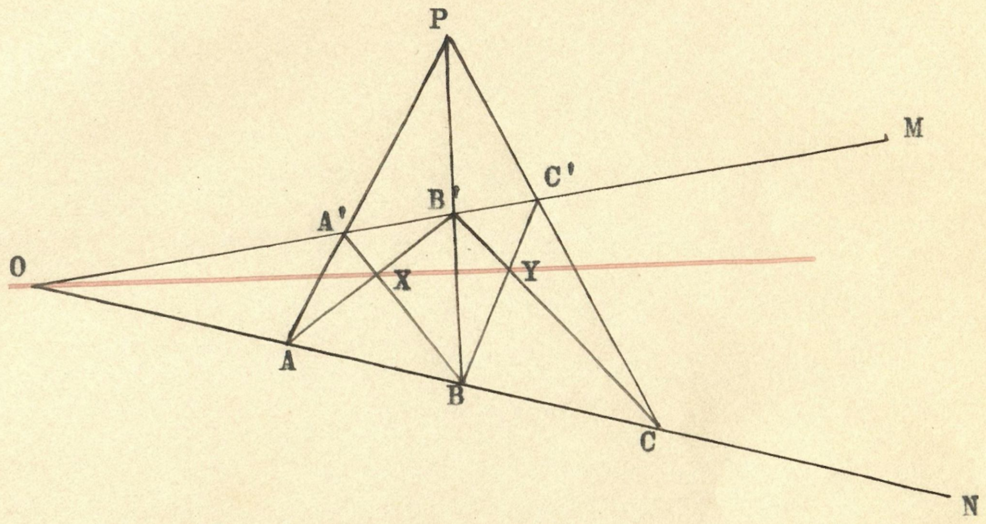
Proof: From O: R(K,L,X,Y) = R(K,J,R,M) (Th.1 & Cor.)
" P: R(K,L,X,Y) = R(J,L,N,S)
Therefore R(K,J,R,M) = R(J,L,N,S) (Equal to same ratio)

But R(K,J,R,M) = R(J,K,M,R) (Cor.1,Th.1)

Therefore R(J,K,M,R) = R(J,L,N,S)

Whence LK (=XY), MN and SR are concurrent by Theorem 2.

THEOREM 7. Given any point P and any two intersecting lines OM and ON; also any three lines PA, PB, PC, through P which cut OM and ON respectively in A'B'C' and ABC. If the diagonals of the two quadrilaterals thus formed be drawn, their intersections, X and Y, are collinear with O.



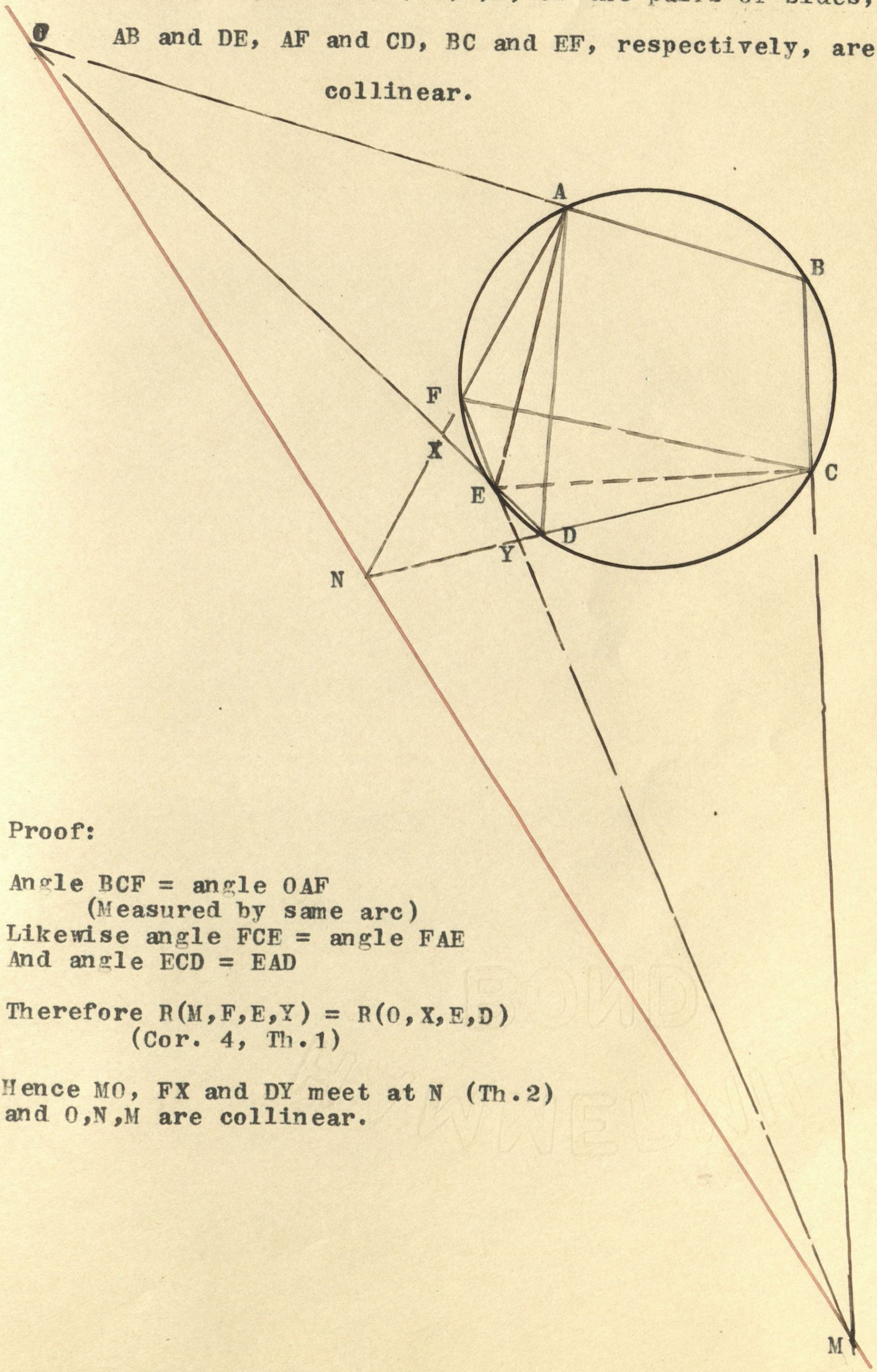
Proof:

The triangles A'BC' and AB'C have AA', BB' and CC' meeting at P by construction.

Therefore the intersections of corresponding sides are on a line by Theorem 5.

Hence O, X, Y are collinear.

THEOREM 8. If the hexagon, ABCDEF, be inscribed in a circle, the intersections, O,N,M, of the pairs of sides, AB and DE, AF and CD, BC and EF, respectively, are collinear.



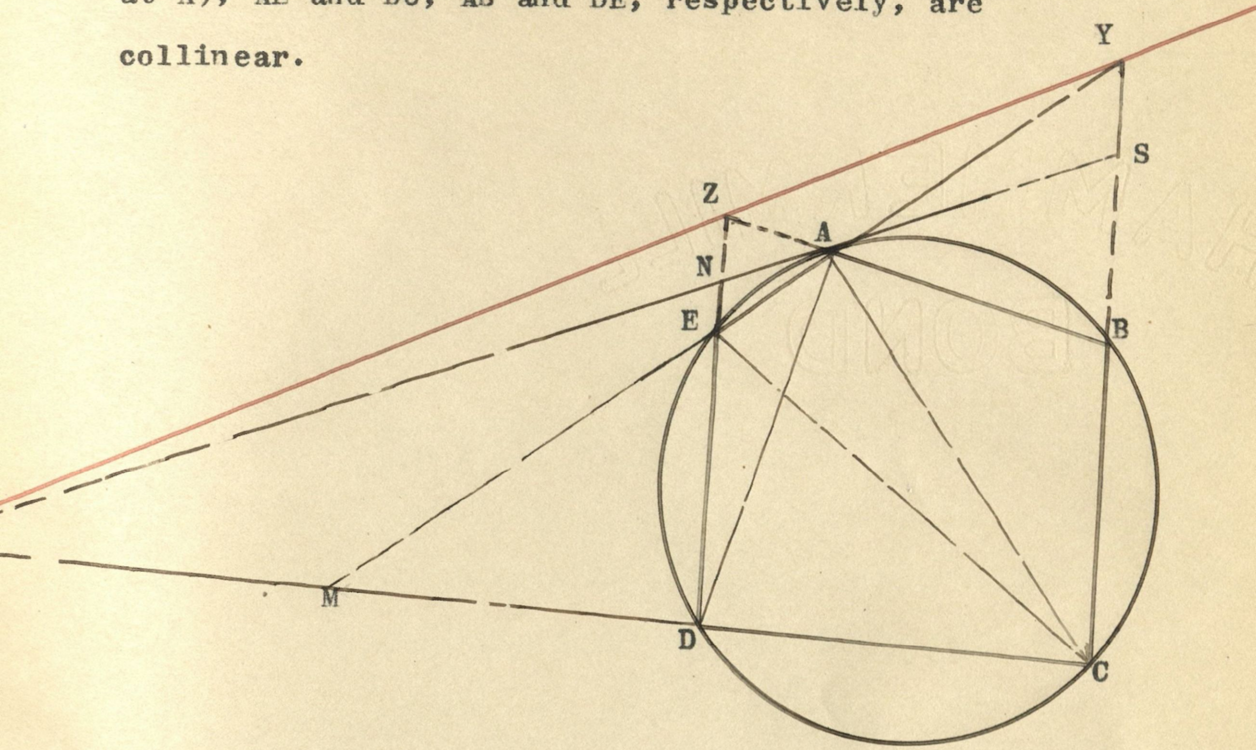
Proof:

Angle BCF = angle OAF
(Measured by same arc)
Likewise angle FCE = angle FAE
And angle ECD = EAD

Therefore R(M,F,E,Y) = R(O,X,E,D)
(Cor. 4, Th.1)

Hence MO, FX and DY meet at N (Th.2)
and O,N,M are collinear.

THEOREM 9. If the pentagon, ABCDE, be inscribed in a circle, the intersections, X,Y,Z, of CD and AN (tangent at A), AE and BC, AB and DE, respectively, are collinear.



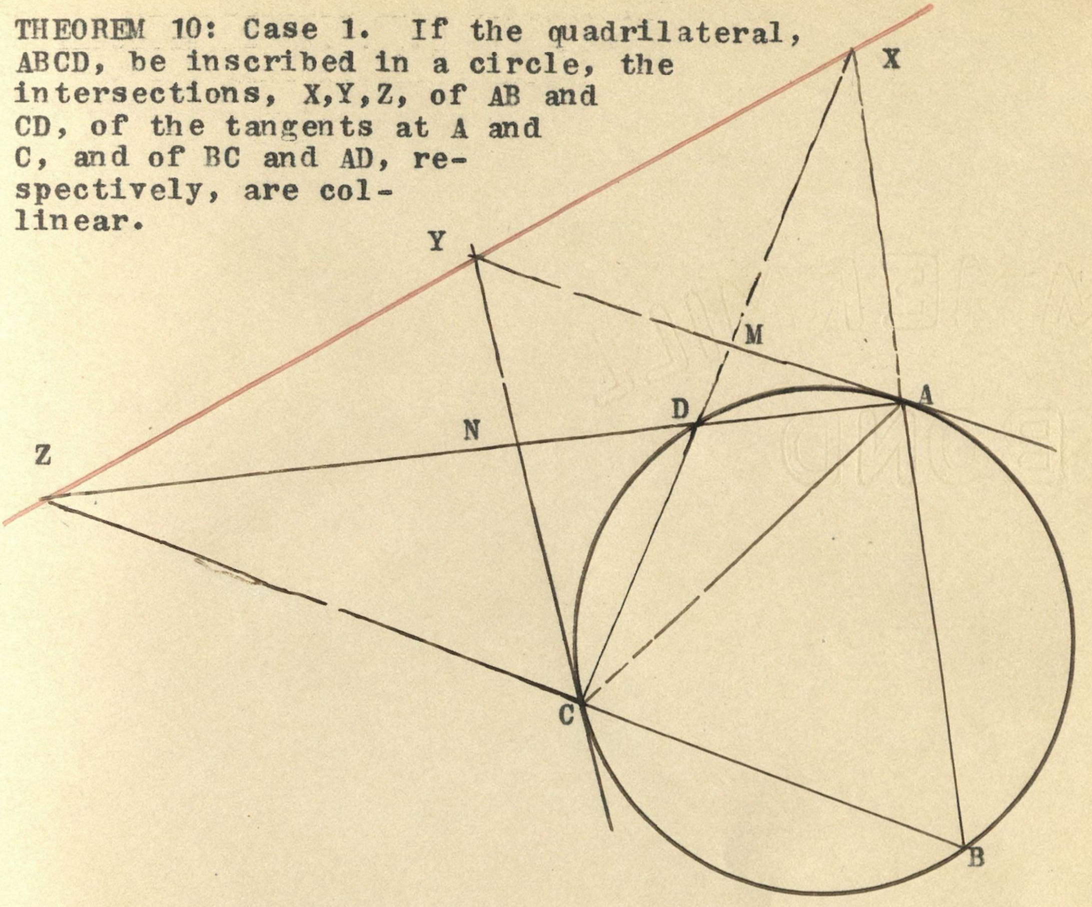
Proof:

Angles ACB, ACE and ECD at C are equal to angles SAB, NAE and EAD at A, respectively.
(Measured by equal arcs in each case)

Hence R(Y,A,E,M) = R(Z,N,E,D) (Cor.4, Th.1)

Therefore YZ, AN and MD (=DC) are concurrent by Theorem 2 and X,Y,Z are collinear.

THEOREM 10: Case 1. If the quadrilateral, ABCD, be inscribed in a circle, the intersections, X,Y,Z, of AB and CD, of the tangents at A and C, and of BC and AD, respectively, are collinear.



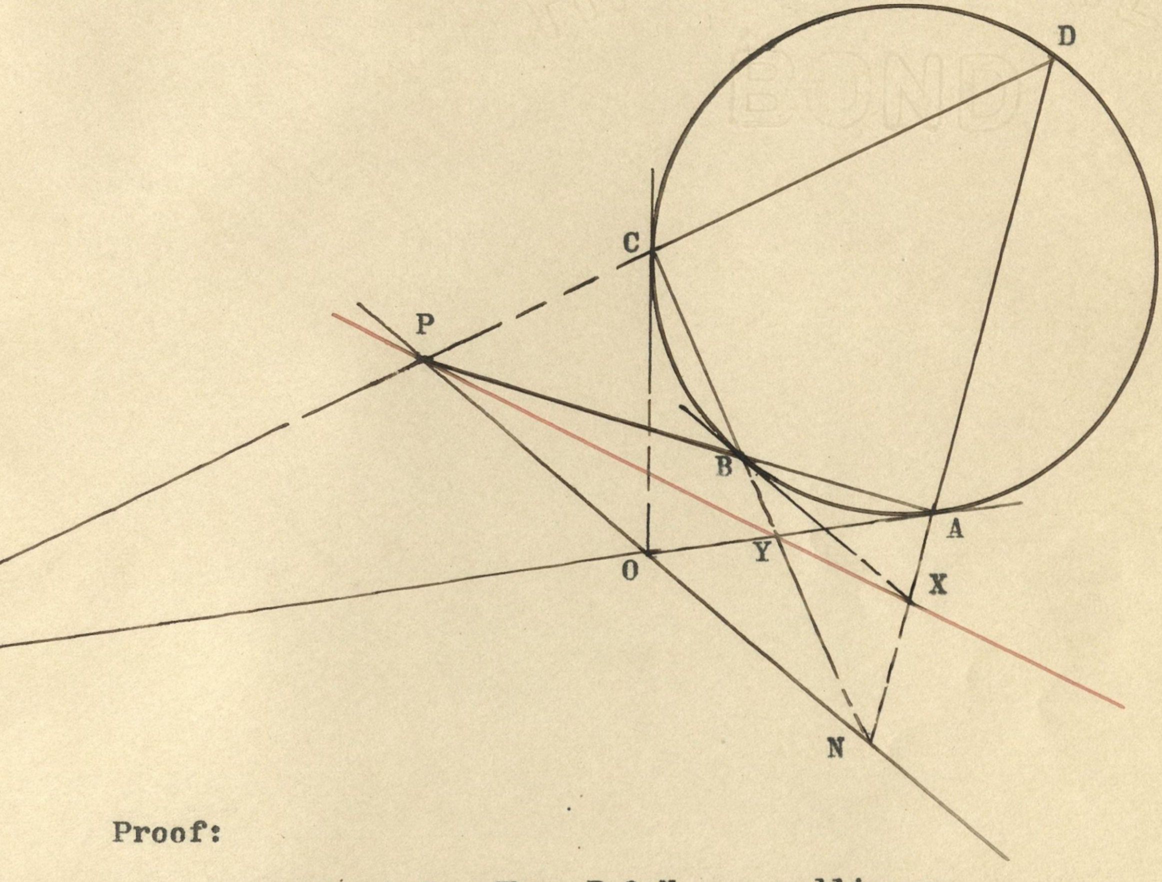
Proof:

Angles XAM, MAD and DAC at A are equal to angles BCA, ACD and DCN at C, respectively.
(Measured by equal arcs)

Hence R(X,M,D,C) = R(Z,A,D,N) (Cor. 4,Th.1)

Therefore XZ, MA and NC are concurrent by Theorem 2 and X, Y and Z are collinear.

THEOREM 10: Case 2. If the quadrilateral, ABCD, be inscribed in a circle, the intersections, X,Y,P, of AD and the tangent at B, of BC and the tangent at A, and of AB and CD, respectively, are collinear.



Proof:

Draw tangent at C. Then P,O,N are collinear by Case 1.

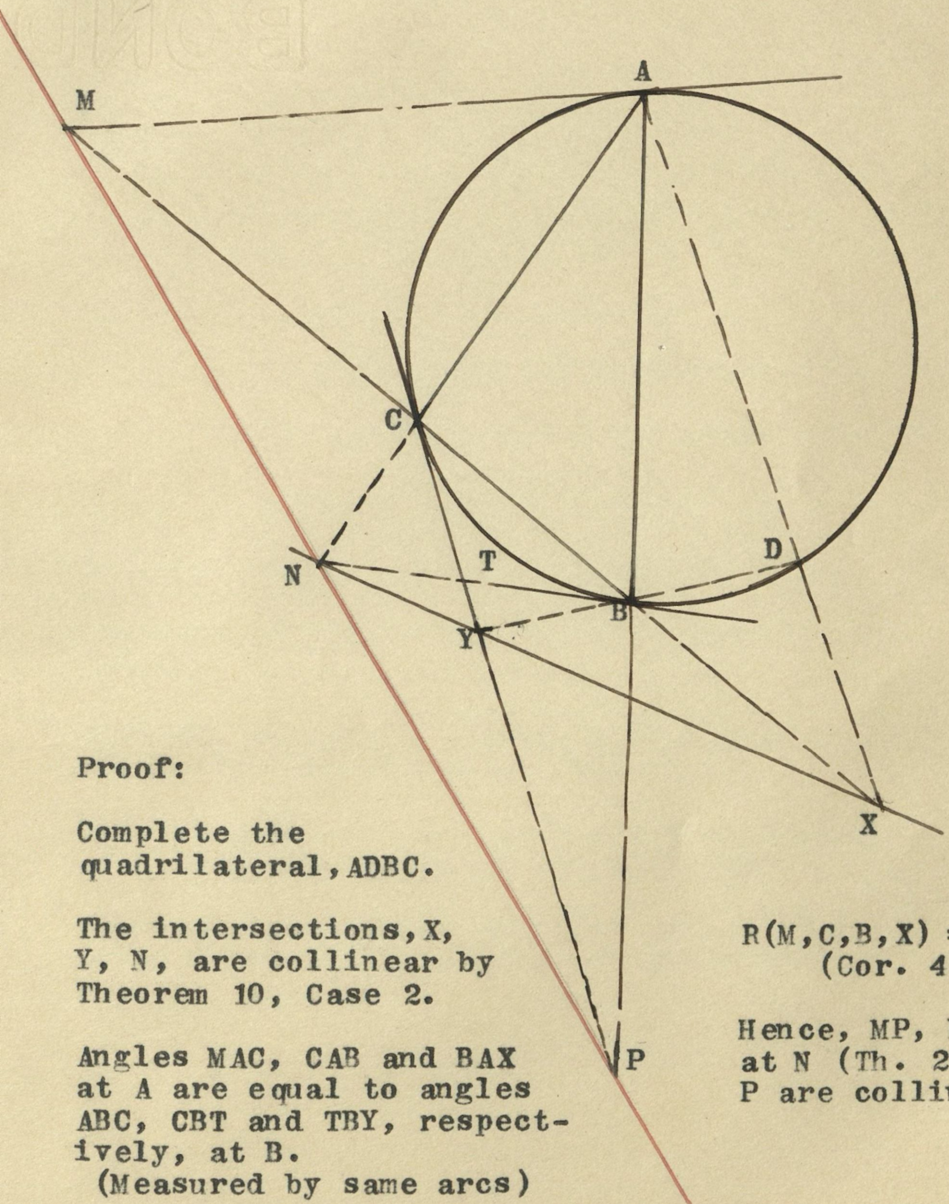
Angles DBA, ABX and XBN at B are equal to angles DCA, ACB and BCO, respectively, at C.
(Measured by equal arcs)
R(D,A,X,N) = R(S,A,Y,O) (Cor. 4, Th. 1)

Therefore DS, XY and NO are concurrent by Theorem 2

and X, Y and P are collinear.

Note: Cases 1 and 2 cover all possible figures as the tangents must be at adjacent or alternate vertices.

THEOREM 11. If the triangle, ABC, be inscribed in a circle, the intersections, M,N,P, of BC and tangent AM, of AC and tangent BN, and of AB and tangent CP, respectively, are collinear.



Proof:

Complete the quadrilateral, ADBC.

The intersections, X, Y, N, are collinear by Theorem 10, Case 2.

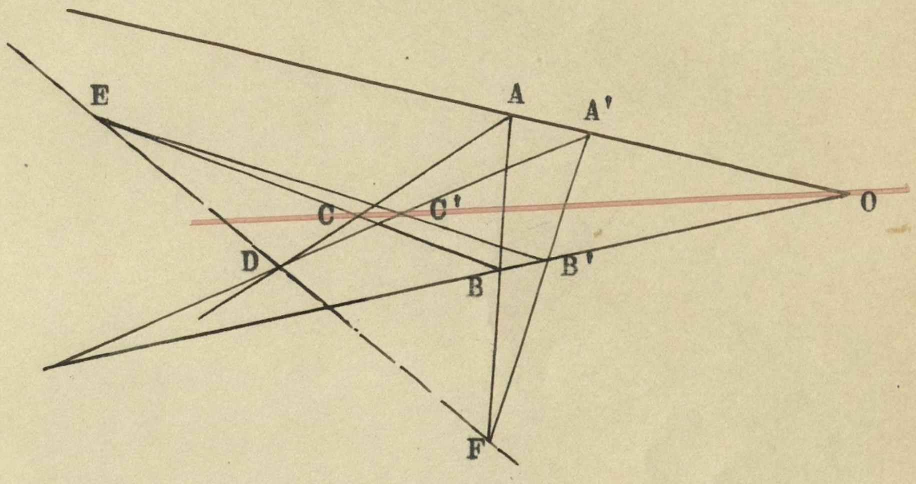
Angles MAC, CAB and BAX at A are equal to angles ABC, CBT and TBY, respectively, at B.
(Measured by same arcs)

R(M,C,B,X) = R(P,C,T,Y)
(Cor. 4, Th.1)

Hence, MP, BT and XY meet at N (Th. 2) and M, N and P are collinear.

EXERCISE

If the three sides of a variable triangle, ABC, rotate about three collinear points, D,E,F, respectively, while two vertices, A and B, move upon two fixed lines which intersect at O, show that the third vertex, C, will describe a straight line through O.



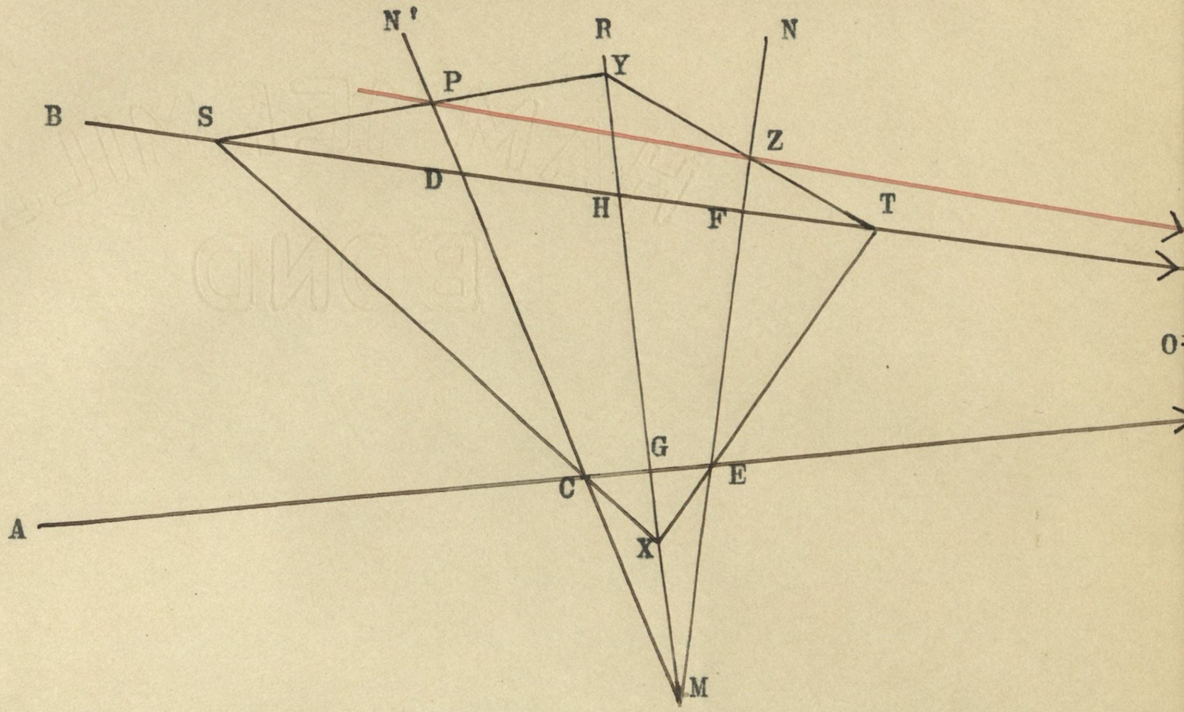
Proof:

Let ABC and A'B'C' be any two positions of the variable triangle with the given conditions.

AC and A'C' meet at E, BC and B'C' meet at D, and AB and A'B' meet at F. E,D,F are collinear by hypothesis.

Therefore the lines AA', BB' and CC' must meet at O, (Th.3) and C must travel on the line OCC'.

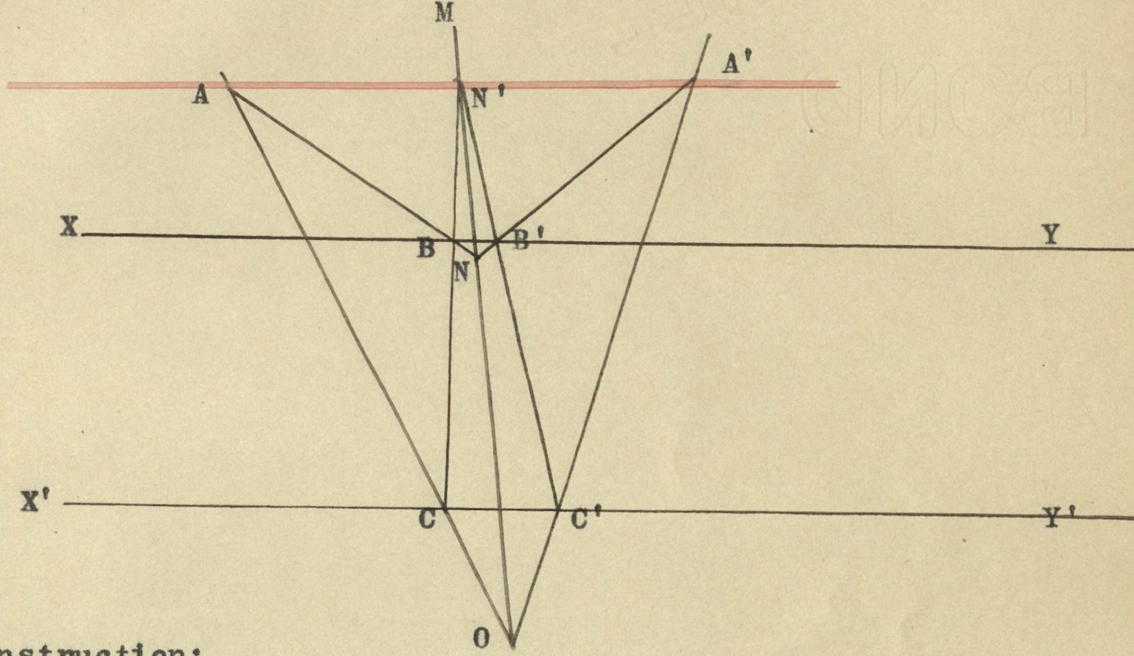
CONSTRUCTION 1. Given OB and OA, meeting at some inaccessible point O, and any point P. To draw a line through P which shall pass through O.



Construction: Draw MN' through P and MN and MR. Let them intersect OA and OB as indicated. From any point, S, on OB, draw SC. Let it cut MR in X. Draw XE and let it cut OB in T. Also draw SP cutting MR in Y. Draw YT. It cuts MN in some point Z. PZ is the desired line.

Proof:
Triangles ZTE and PSC have their respective sides meeting in pairs on the line MR. Therefore, the lines, PZ, TS and EC must be concurrent by Theorem 3. But TS and EC meet at O by hypothesis. Therefore, PZ must also pass through that point.

CONSTRUCTION 2. Through a given point, A, to draw a line parallel to two given lines which are parallel to each other. Let XY and X'Y' be the given lines.



Construction:
Through any point, O, draw OA and any two lines, OM and OA'. Draw any line from A cutting XY in B and OM in N. Draw CB and let it cut OM in N'. Draw N'C' cutting XY in B'. Draw NB' cutting OA' in A'. AA' is the required line.

Proof: Since the sides of triangle ABC each meet a side of triangle A'B'C' in one of the three collinear points, N'NO, the lines AA', BB' and CC' must be concurrent if they meet at all. But BB' and CC' are parallel by hypothesis. Since they cannot meet, it follows that AA' cannot meet either of them and must be parallel to them.

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